Comput. Math. Applic. Vol. 15, No. 2, pp. 85–96, 1988
Printed in Great Britain.

0097-4943/88 $3.00 + 0.00

A MATHEMATICAL MODEL FOR THE BIOMECHANICS OF THE BODY OF A VERTEBRA

J. C. Misra and S. Samanta
Department of Mathematics, Indian Institute of Technology, Kharagpur 721-302, India

(*Received June* 1987)

Communicated by E. Y. Rodin

Abstract—A mathematical model is developed in the paper for studying the dynamical behaviour of the vertebral body of a vertebra. In accordance with experimental observations, the three principal mechanical properties of bone tissues, viz. material damping, anisotropy and inhomogeneity, have been paid due attention. Frequency spectra and damping coefficients are computed for two different types of vibratory motion of a specific bone specimen representing the vertebral body of a vertebra.

1. INTRODUCTION

Biomechanics was born as a new separate field of interdisciplinary research to meet the requirements of human society with an aim to derive in details the data characterizing human organisms and to use them for the purpose of descriptions of phenomena taking place in human bodies. The coverage of problems is very wide-ranging, from mechanical properties of the microstructure over the problems of the vascular system and the motion systems to the design of artificial external and internal replacement of parts of a living organism.

In this paper, our specific interest is to explore some informations regarding the dynamical behaviour of the vertebral body of a vertebra. It is known that in all vertebrate animals the central axis of the body consists of a vertebral column. As it is essential that provision should be made for a considerable range of movement of the trunk, the column consists, not of a single elongated bone, but of a number of independent, irregular bones, termed the vertebrae which are firmly connected to one another. The provision of a central axis is not the only function which the column has to subserve. It is built up so as to surround the spinal cord (cf. Fig. 1), to which it affords

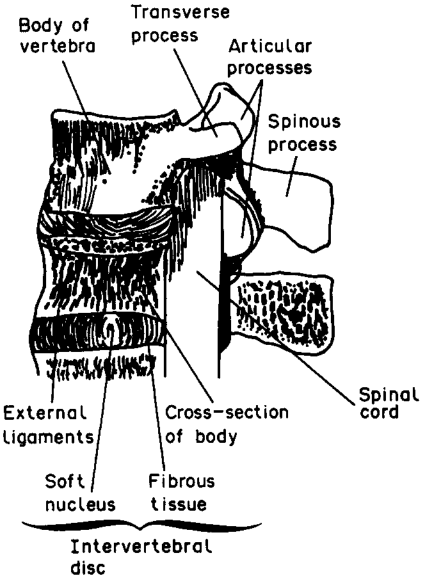

Fig. 1. Body of vertebra, intervertebral disc and the spinal cord.

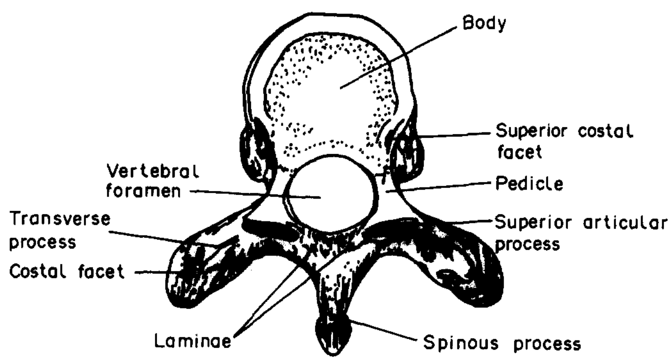

Fig. 2. A typical thoracic vertebra, superior aspect.

necessary protection. The human vertebral column must also support the weight of the trunk and transmit it to the lower limbs. The vertebrae are grouped under the names cervical, thoracic, lumbar, sacral and coccygeal or caudal, according to the region in which they lie, but all conform to a general plan not only in man but also in all other vertebrate animals [1].

A typical vertebra (cf. Fig. 2) consists of two principal parts, the anterior or ventral part is called the body, while the posterior or dorsal part is known as the vertebral foramen. The opposed surfaces of the bodies of adjoining vertebrae are firmly connected to each other by discs of fibrocartilage, termed intervertebral discs (Fig. 1), each of which acts like a cushion, flattening under pressure and is also able to shift its position to accommodate to the motions of the vertebrae as they twist and turn and as the whole column bends. These discs allow movement of the vertebrae and act as buffers to the spinal column against the shock of walking or jumping.

The body of a vertebra is more or less cylindrical, but is subject to a wide range of variation in size and shape in different animals and in different regions of the same animal [1]. The vertebral body is composed of spongy substance covered by a thin coating of compact bones. On the upper and lower surfaces of the body the coating of compact bone is thinner than elsewhere, but in the vertebral arch and the processes projecting from it, as shown in Fig. 1, it is considerably thicker than it is in the body.

During daily activities of life, human and subhuman primates are often exposed to vibration environments. Also, when a subject is placed on a vibration table, whole-body accelerations (vibrations) are instantaneously induced in the body. For the treatment of cardiogenic or anoxic shock, as a therapeutic measure, sometimes whole-body accelerations are imparted to the human body. Further, the jackhammer operators, helicopter crew members, astronauts riding boosters and many others have to carry on their professional activities in vibration environments. During such activities their bodies are subjected to various combinations of amplitudes and frequencies, which should be thoroughly examined and standards should be established for long-term as well as short-term exposures to vibration environments. It is worthwhile to point out here that even less dramatic vibration effects may sometimes be more serious. Moreover, when a man drives a car or flies in an aircraft or spacecraft, he is often subjected to unintentional body accelerations. In view of all these, vibration physiology is of serious concern in the space and military aircraft fields too. Also, owing to the clinical importance in the treatment of acute and spinal disorders, studies on the intervertebral motion of human spine have generated significant interest amongst researchers in the field.

Quandieu and Pellieux [2] carried out an experimental study dealing with the *in vivo* measurements of the vibrational accelerations in the lumbar spine of a restrained and conscious, sitting baboon. This work provides valuable information for modelling the spine, as well as for developing diagnostic techniques to evaluate the functional condition of the spine. Evans [3] discussed the tensile and compressive strain of the vertebral column and sacral alae measured in fully restrained human cadavers during controlled acceleration. Soni *et al.* [4] described a spine fixture and linkage transducer used to measure kinetics of ten lumbar spine segments. They presented data on the range of vertebral motion, its components and parameters associated with the screw motion. While studying the bulging of lumbar intervertebral discs experimentally, Reuber *et al.* [5] reported that the vertebral body horizontal translations were sometimes as large as, and on occasions larger than, the largest transverse bulges. Gross motion of the vertebral body was also studied through different experimental investigations by Lin *et al.* [6, 7] and Nachemson *et al.* [8].

However, investigators have often experienced that it is difficult to get the complete picture of the stress and strain fields of the vertebral body and the associated components of the vertebra through experimental investigations. On the other hand, mathematical model analyses together with the process of parametric variation possess the potential to explore a variety of informations regarding the influences of the geometry and material property variations. Recently several model studies of some associated problems have been made by various investigators (see [9, 10]). In the case of model studies, however, extensive studies of the specimen response are possible only when accurate experimental data for the mechanical properties of the specimen are used and secondly when the model is simple. It may be mentioned in this connection that many of the finite element models analysed by previous researchers have been found to be not very useful for comprehensive parameter studies.

Keeping these two points in view, an attempt is made here to put forward a mathematical analysis for the dynamical behaviour of the vertebral body, by paying due attention to the important characteristics of bone tissues, ascertained through various experimental investigations mentioned below.

Studies of different specimens of the skeletal system assert that bone is a composite material, one phase being represented by the viscoelastic bonding and the other by the osteons. McElhaney [11] carried out experiments on different specimens of bone and reported that bone tissues exhibit viscoelastic properties of creep and relaxation. Lakes *et al.* [12] ascertained the viscoelastic properties of wet cortical bones. Subsequently, the viscoelastic material behaviour of bones on the basis of their microstructure was analysed by Gottesman and Hashin [13]. In recent years, the viscoelastic behaviour of collagen which is one of the principal constituents of bone tissues, was analysed experimentally by Sanjibee [14] as well as by Sanjibee *et al.* [15]. By considering the viscoelastic properties of the bone tissues, Spilker [16] as well as Spilker *et al.* [17] studied the mechanical response of the intervertebral discs under the action of complex loading, by using finite element technique. An intervertebral disc was modelled by them as a circular disc that lumps the cortical and trabecular bone regions together with the body and the cartilaginous end plates (cf. Fig. 1). While studying the creep behaviour of the intervertebral discs, Burns and Kaleps [18] as well as Burns *et al.* [19] examined the two-, three- and four-parameter Kelvin models as possible descriptions of the material damping behaviour of the vertebral discs.

Lang [20] measured experimentally the anisotropic elastic moduli of bone tissues, while Yoon and Katz [21] through an analytical study confirmed that osseous tissues possess hexagonal characteristics. Recently while carrying out an experiment on bone material by employing extensional wave technique, Lipson and Katz [22] observed that plexiform bone, in which all the three orthogonal directions are structurally distinguishable, is orthotropic while Haversian bone is transversely isotropic.

Nowinski [23], while studying the stress-field in a specimen of bone, considered the radial inhomogeneity of osseous tissues. But his analysis was restricted to merely an ideal situation in which all the material properties of the osseous material vary in an identical manner according to a power law. In fact he regarded all the material parameters as a power function of the radial distance. But Patel [24] predicted that for porous materials like bones, the elastic modulus is related to the apparent density according to a power law. This was ascertained recently by Carter *et al.* [25] and also by Stone *et al.* [26] who carried out experiments on different specimens of bone.

The problem of free vibration of the cranial vault was studied by Misra and Chakravarty [27] by taking into account the dissipative material behaviour of both the skull and the brain. They reported that the material damping of both the skull and the brain affect the frequency spectrum of the freely vibrating cranial vault quite significantly. The material inhomogeneity and anisotropy of osseous tissues were incorporated by Misra and Samanta [28] in their study of a torsional problem of tubular bones. They examined the effects of material damping and inhomogeneity on the attenuation and phase velocities of the waves propagating in a bone specimen. Misra and Samanta [29] also derived the dispersion relation for axisymmetric acoustic waves propagating along the axis of a long composite bone, by employing an analytical procedure. They remarked that the viscoelastic and piezoelectric effects in the vibrating tubular bone specimen might be looked upon as small perturbation effects to a general elastic material behaviour of the bone structure. Very recently the same authors [30] made a thorough investigation on the effect of material damping on the physiological process of bone remodelling.

The purpose of the present paper is to develop and analyse a mathematical model for studying two different types of vibratory motion of the body of the vertebra. The motion is considered to be axially symmetric about the longitudinal axis of the bone shaft. The derived results, it is believed, will find useful applications in correlating the mechanical properties of the vertebral body with the vibration parameters. A similar model for the axisymmetric vibration of a bone specimen approximated as a hollow cylinder is also analysed. The results of this part of the study will be particularly useful when one aims at correlating the observations of the vibration experiment performed on a long tubular bone (e.g. femur, tibia etc.).

Considering the anisotropic relaxation functions for the bone tissues as independent of frequency at the ultrasound range [12], the frequency spectra as well as the damping coefficients are computed for a given specimen of bone. A second possibility of accounting for the material damping behaviour of osseous tissues, has also been examined where the bone medium is treated as a standard linear solid. The results of numerical computation are presented in tabular form. The computed values clearly indicate that the vibration characteristics of the bone specimens studied here are significantly influenced by inhomogeneity as well as material damping of osseous tissues.

2. BASIC EQUATIONS

Let (r, θ, z) be the cylindrical coordinates of a representative point of a given specimen of the vertebral body (cf. Fig. 3). We denote by $(e_{rr}, e_{\theta\theta}, e_{zz}, e_{\theta z}, e_{rz}, e_{r\theta})$ the six components of the strain tensor at the point and by $(\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}, \tau_{\theta z}, \tau_{rz}, \tau_{r\theta})$ those of the stress tensor. Considering the anisotropy and material damping of osseous tissues, the constitutive relations for the vertebral body may be put as

$$\begin{aligned}
\tau_{rr} &= C^0_{11}(D)e_{rr} + C^0_{12}(D)e_{\theta\theta} + C^0_{13}(D)e_{zz} \\
\tau_{\theta\theta} &= C^0_{12}(D)e_{rr} + C^0_{11}(D)e_{\theta\theta} + C^0_{13}(D)e_{zz} \\
\tau_{zz} &= C^0_{13}(D)e_{rr} + C^0_{13}(D)e_{\theta\theta} + C^0_{33}(D)e_{zz} \\
\tau_{r\theta} &= C^0_{66}(D)e_{r\theta} = \tfrac{1}{2}[C^0_{11}(D) - C^0_{12}(D)]e_{r\theta} \\
\tau_{\theta z} &= C^0_{44}(D)e_{\theta z} \quad \text{and} \quad \tau_{rz} = C^0_{44}(D)e_{rz},
\end{aligned} \tag{1}$$

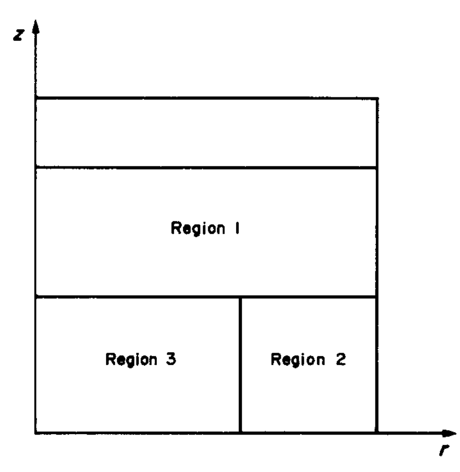

Fig. 3. Simplified model of the vertebral body/intervertebral disc. Regions: (1) vertebral body; (2) annulus fibrosis; (3) nucleus pulposis.

in which $C^0_{IJ}(D)$ $(I, J = 1, 2, 3)$ are functions of the partial differential operator $D = \partial/\partial t$ and represent viscoelastic relaxation operators. In order to take the radial inhomogeneity of osseous tissues into account (see Refs [24, 25]), let us write

$$C^0_{IJ}(D) = K_{IJ}(D)(\rho^0)^m, \tag{2}$$

in which ρ^0 represents the apparent density, that is the mass per unit volume of the bone specimen excluding voids, $K_{IJ}(D)$ are some functions of D and m is a parametric constant.

If we consider the following power law for the variation of density:

$$\rho^0 = \rho r^{p+\alpha} \tag{3}$$

(α being a parametric constant and ρ being independent of r), we have

$$C^0_{IJ}(D) = C_{IJ}(D) r^p, \tag{4}$$

where

$$C_{IJ}(D) = \rho^m K_{IJ}(D)$$

and p is related to m as

$$m = \frac{p}{p + \alpha}.$$

It may be noted that $C_{IJ}(D)$ serve as the relaxation operator when $p = 0$. In the particular case, when the elastic moduli vary linearly with the apparent density ρ^0, which is almost true for shear moduli [24], m is equal to unity; this implies further that $\alpha = 0$ for all values of p.

The kinematic relations that connect the components of the displacement vector and those of the strain tensor may be put in the form

$$e_{rr} = \frac{\partial U}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\left(\frac{\partial V}{\partial \theta} + U\right), \quad e_{zz} = \frac{\partial W}{\partial z}, \quad e_{\theta z} = \frac{\partial V}{\partial z} + \frac{1}{r}\frac{\partial W}{\partial \theta}, \quad e_{zr} = \frac{\partial W}{\partial r} + \frac{\partial U}{\partial z}$$

$$\text{and} \quad e_{r\theta} = \frac{1}{r}\left(\frac{\partial U}{\partial \theta} - V\right) + \frac{\partial V}{\partial r}, \tag{5}$$

in which U, V and W are components displacement along radial, circumferential and axial directions respectively.

The equations of motion, in cylindrical polar coordinates, are given by

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = \rho^0 \frac{\partial^2 U}{\partial t^2}$$

$$\frac{\partial \tau_{r\theta}}{\delta r} + \frac{1}{r}\left(\frac{\partial \tau_{\theta\theta}}{\partial_\theta} + 2\tau_{r\theta}\right) + \frac{\partial \tau_{\theta z}}{\partial z} = \rho^0 \frac{\partial^2 V}{\partial t^2}$$

and

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\left(\frac{\partial \tau_{\theta z}}{\partial \theta} + \tau_{rz}\right) + \frac{\partial \tau_{zz}}{\partial z} = \rho^0 \frac{\partial^2 W}{\partial t^2}. \tag{6}$$

3. METHOD OF SOLUTION

Let us consider the vibration of the vertebral body vibrating with circular frequency ω defined by the relations

$$U(r, t) = u(r)e^{i\omega t}, \quad V = 0, \quad W = 0. \tag{7}$$

Under such a situation, the second and third equations of (6) are identically satisfied. Using equations (1), (3)–(5) and (7), we obtain the following second order partial differential equation from the first of the equation (6) with $\tau_{\theta r} = 0$:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1+p}{2}\frac{\partial u}{\partial r}+\left(\frac{K^2}{r^{-\alpha-2}}-m'^2\right)\frac{u}{r^2}=0, \tag{8}$$

where

$$K^2=\frac{\rho\omega^2}{C_{11}(i\omega)} \quad \text{and} \quad m'^2=\frac{C_{11}(i\omega)-pC_{12}(i\omega)}{C_{11}(i\omega)} \tag{9}$$

(C_{IJ} are now functions of $i\omega$, $i=\sqrt{-1}$).

Substituting $u=r^{-p/2}f(r)$ and $z=[2K/(\alpha+2)]r^{(\alpha+2)/2}$ [$f(r)$ being a function of r alone] in (9), one obtains

$$\frac{\partial^2 f}{\partial z^2}+\frac{1}{z}\frac{\partial f}{\partial z}+\left(1-\frac{m^2}{z^2}\right)f=0, \tag{10}$$

in which

$$m^2=\left(\frac{C_{11}-pC_{12}}{C_{11}}+\frac{p^2}{4}\right)\Big/\left(\frac{\alpha+2}{2}\right)^2. \tag{11}$$

The general solution of equation (10) is given by

$$u(r)=r^{-p/2}[AJ_m(z)+BY_m(z)], \tag{12}$$

A and B being two arbitrary constants of integration and $J_m(z)$, $Y_m(z)$ being ordinary Bessel functions of the first and the second kind respectively, each of order m.

In the case of a vertebral body of radius, $r_0(say)$, the displacement $u(r)$ given by (12) is finite at $r=0$, only when $p\leqslant 0$ and $B=0$. Thus in this case, the general expression for the radial displacement has to be taken as

$$u(r)=Ar^{-p/2}J_m(z), \quad (p\leqslant 0). \tag{13}$$

For free vibration of the bone specimen under consideration, the outer surface of the specimen is traction-free, so that

$$\tau_{rr}=0 \quad \text{at} \quad r=r_0. \tag{14}$$

Using (13) in (15), one obtains the frequency equation

$$C_{11}Kr_0^{\alpha/2}J'_m(z_0)+\frac{C_{12}-\frac{1}{2}pC_{11}}{r_0}J_m(z_0)=0, \tag{15}$$

where

$$z_0=\frac{2K}{\alpha+2}r_0^{(\alpha+2)/2}. \tag{16}$$

Here and in the sequel, a prime over a function denotes differentiation with respect to the argument of the function. If the surface of the bone specimen be clamped,

$$u(r_0)=0, \tag{17}$$

so that the frequency equation reduces to

$$J_m(z_0)=0. \tag{18}$$

Let us now consider a tubular bone specimen having the shape of a hollow circular cylinder. If r_i and r_0 be the endosteal and periosteal radii of such a specimen, for free vibration, we have

$$\tau_{rr}=0 \quad \text{at} \quad r=r_i \quad \text{and} \quad r=r_0. \tag{19}$$

In this case, the frequency equation reads

$$[J'_m(z_0)+M_0J_m(z_0)][Y'_m(z_i)+M_iY_m(z_i)]=[J_m(z_i)+M_iJ_m(z_i)][Y'_m(z_0)+M_0Y_m(z_0)], \tag{20}$$

in which

$$M_i = \frac{C_{12} - \frac{1}{2}pC_{11}}{C_{11}Kr_i^{\alpha/2+1}}, \quad M_0 = \frac{C_{12} - \frac{1}{2}pC_{11}}{C_{11}Kr_0^{\alpha/2+1}}, \tag{21}$$

$$z_i = \frac{2K}{\alpha+2} r_i^{(\alpha+2)/2} \quad \text{and} \quad z_0 = \frac{2K}{\alpha+2} r_0^{(\alpha+2)/2}. \tag{22}$$

In case, the inner and outer surfaces of the specimen of bone are clamped, we have

$$u(r_0) = 0 \quad \text{and} \quad u(r_i) = 0. \tag{23}$$

From (12) and (23), the frequency equation for such a situation is obtained as

$$J_m(z_0)Y_m(z_i) = J_m(z_i)Y_m(z_0). \tag{24}$$

4. A SECOND TYPE OF VIBRATION

Another possible type of vibration for the vertebral body specimen, may be defined by the relations

$$U = 0, \quad V = v(r)e^{i\omega t}, \quad W = 0. \tag{25}$$

Using (25), (5) and (1) in the only non-trivial equation of motion,

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = \rho^0 \frac{\partial^2 V}{\partial t^2}, \tag{26}$$

and taking care of the bone inhomogeneity described by (3) and (4), we obtain

$$\frac{\partial^2 v}{\partial r^2} + \frac{1+p}{r}\frac{\partial v}{\partial r} + \left(\frac{K'^2}{r^{-\alpha-2}} - \mu'^2\right)\frac{v}{r^2} = 0, \tag{27}$$

in which

$$K'^2 = \frac{\rho\omega^2}{C_{66}}, \tag{27a}$$

and $\mu'^2 = 1 + p$, C_{66} being considered as a function of $i\omega$.

The general solution of (27) is given by

$$v(r) = r^{-p/2}[A_1 J_\mu(z) + B_1 Y_\mu(z)], \tag{28}$$

where

$$\mu^2 = (1 + p + p^2/4) \Big/ \left(\frac{\alpha+2}{2}\right)^2, \quad z = \frac{2K'}{\alpha+2} r^{(\alpha+2)/2},$$

A_1 and B_1 are arbitrary constants.

For a solid bone structure, in order to avoid singularity at $r = 0$, the solution (28) has to be taken as

$$v(r) = r^{-p/2} A_1 J_\mu(z), \tag{29}$$

where $p \leqslant 0$.

For shear-free motion, we must have

$$\tau_{r\theta} = 0 \quad \text{at} \quad r = r_0. \tag{30}$$

Using (29) in (30), one obtains the frequency equation

$$K' r_0^{\alpha/2} J'_\mu(z_0) - \frac{p+2}{2r_0} J_\mu(z_0) = 0, \tag{31}$$

where

$$z_0 = \frac{2K'}{\alpha+2} r_0^{(\alpha+2)/2}.$$

For free vibrations of a hollow tubular bone structure, we must have

$$\tau_{r\theta} = 0 \quad \text{at} \quad r = r_i \quad \text{and} \quad r = r_0. \tag{32}$$

Proceeding as before, the following frequency equation is derived:

$$[J'_\mu(z_0) - N_0 J_\mu(z_0)][Y'_\mu(z_i) - N_i Y_\mu(z_i)] = [J'_\mu(z_i) - N_i J_\mu(z_i)][Y'_\mu(z_0) - N_0 Y_\mu(z_0)], \tag{33}$$

in which

$$N_i = \frac{p+2}{K' r_i^{(\alpha/2)+1}}, \quad N_0 = \frac{p+2}{K' r_0^{(\alpha/2)+1}}, \quad z_i = \frac{2K'}{\alpha+2} r_i^{(\alpha+2)/2} \quad \text{and} \quad z_0 = \frac{2K'}{\alpha+2} r_0^{(\alpha+2)/2}. \tag{34}$$

5. NUMERICAL ILLUSTRATION

In carrying out the numerical calculations, we have considered two viscoelastic models. In the first model, the relaxation functions are assumed to be independent of frequency in the ultrasound range and the value of the loss tangent is taken to be 0.01 [12], so that

$$C_{IJ} \equiv C_{IJ}(1 + 0.01i) \tag{35}$$

in which $i = \sqrt{-1}$ and C_{IJ} are the material parameters. For the purpose of comparison, a second viscoelastic model—the standard linear solid model for bone [18]—is also used. Generalizing this model for anisotropic material, one can write

$$C_{IJ}(i\omega) \equiv C_{IJ}\left(\frac{1 + i\omega T_1}{1 + i\omega T_2}\right), \tag{36}$$

where T_1 and T_2 are the model parameters.

Utilizing expression (34) or (35) and the known values of the material parameters, the frequency equations (15), (18), (20), (24), (31) and (33) are solved to obtain the frequencies and the damping co-efficients for solid and hollow tubular bone structures.

6. VERTEBRAL BODY

The following values of the material constants have been used for the computational work:

$$C^0_{11} = C^0_{22} = C^0_{33} = 18.96\,\text{MPa}; \quad C^0_{12} = C^0_{13} = C^0_{66} = 6.3\,\text{MPa}$$

(since the requisite data for the anisotropic elastic moduli are not known, average isotropic values have been used in this part of the computational work) and

$$\rho^0 = 2000\,\text{kg/m}^3.$$

The above data are supposed to be the representative values at $r = 0.02$m (i.e. the outer surface of the vertebral body). The loss tangent for the first model has been taken to be 0.01 (as mentioned earlier), whereas the values of the time parameters T_1 and T_2 for the second model are taken as

$$T_1 = 10^4\,\text{s} \quad \text{and} \quad T_2 = 5 \times 10^4\,\text{s}.$$

Now from (9), we have

$$K^2 = \frac{\rho^0 \omega^2}{C^0_{11} r_0^\alpha},$$

so that

$$\omega = K\left[\frac{r_0^\alpha}{\rho^0} C^0_{11}(i\omega)\right]^{1/2}.$$

By considering ω to be a complex quantity represented by $\omega' + i\omega''$, in which the frequency appears as its real part and the damping coefficient as the imaginary part. Equation (36) is utilized to obtain the values of the frequency and the damping coefficient for the vibration, free as well as clamped. Both representations (34) and (35) have been used. A similar procedure has been followed for the calculation of the vibration parameters of the vertebral body specimen for the second type of vibration. Both the procedures are repeated for different sets of values of the inhomogeneity parameters, p and α.

7. HOLLOW TUBULAR BONE SHAFT

In order to account for the anisotropic character of a long tubular bone, this part of the computational work has been carried out by taking (see Ref. [20])

$$C_{11} = 2.38, \quad C_{33} = 3.34, \quad C_{12} = 1.02, \quad C_{66} = 0.68$$

(all in $10^{10}\,\mathrm{N/m^2}$) and $\rho = 2000\,\mathrm{kg/m^2}$. The endosteal and periosteal radii of the concerned bone specimen are taken to be 27.4 and 33.0 mm. The remaining material parameters necessary for the numerical computation are identical to those mentioned earlier. A method similar to that described in the preceding section has been employed to obtain the vibration characteristics of the hollow bone specimen for the two different types of vibratory motions considered in the present analysis, under different conditions of excitation as well as for different types of inhomogeneity and material damping behaviour.

The results of numerical computation depicting the various vibration characteristics of the bone specimens are presented in tabular form in Tables 1–6.

Tables 1–6. Results of numerical calculation for frequencies and damping—coefficients under different conditions of excitations. In the tables, (A) refers to the standard linear solid model for bone, (B) refers to the viscoelastic model with loss-tangent, $\delta = 0.01$, (I) refers to the homogeneous bone medium with $p = 0$ and $\alpha = 0$, (II) refers to the inhomogeneous bone medium with $p = -1$ and $\alpha = 0$, (III) refers to the inhomogeneous medium with $p = -1$ and $\alpha = 0.394$. All the frequencies and the damping coefficients are measured in 10^4 rads/s

Table 1. Values of frequencies (ω') and damping coefficients (ω'') of the vertebral body for free vibration (first type)

		First mode	Second mode	Third mode
I	ω'A	0.3279038	0.7118318	1.107676
	ω''A	0.02043	0.0443607	0.0690332
	ω'B	1.254116	2.7223506	4.2364705
	ω''B	0.00627	0.0136117	0.00211823
II	ω'A	0.3771505	0.7630373	1.159493
	ω''A	0.023505	0.0475544	0.0722626
	ω'B	1.4424678	2.9183489	4.434652
	ω''B	0.00712	0.014917	0.0221623
III	ω'A	0.4047583	0.8646098	1.3381446
	ω''A	0.0252255	0.0538636	0.0833966
	ω'B	1.5480883	3.3055344	5.1179314
	ω''B	0.0074	0.0165276	0.0255896

Table 2. Values of frequencies (ω') and damping coefficients (ω'') of the vertebral body for clamped vibration (first type)

		First mode	Second mode	Third mode
I	ω'A	0.4884424	0.8743036	1.2968504
	ω''A	0.0300876	0.0550882	0.0798848
	ω'B	1.8653778	3.4153748	4.9527143
	ω''B	0.009327	0.0170768	0.0247635
II	ω'A	0.5522489	0.9527197	1.3531902
	ω''A	0.034018	0.0586866	0.0833552
	ω'B	2.109057	3.6384664	5.1678779
	ω''B	0.0105452	0.0181923	0.0258393
III	ω'A	0.5846655	1.0465414	1.5523299
	ω''A	0.0288136	0.0659406	0.1096126
	ω'B	1.786387	4.0882036	5.928399
	ω''B	0.008932	0.020441	0.029642

Table 3. Frequencies (ω') and damping coefficients (ω'') of the vertebral body for free vibration (second type)

		First mode	Second mode	Third mode
I	$\omega'A$	0.3774113	0.6185741	0.8539298
	$\omega''A$	0.0232823	0.0381595	0.0526785
	$\omega'B$	1.4434652	2.3658282	3.2659809
	$\omega''B$	0.007217	0.0118291	0.0163299
II	$\omega'A$	0.3302152	0.5677206	0.804058
	$\omega''A$	0.02037	0.0350224	0.049602
	$\omega'B$	1.262957	2.1713316	3.07524
	$\omega''B$	0.006315	0.0108566	0.05376
III	$\omega'A$	0.146641	0.4117278	0.6850421
	$\omega''A$	0.00905	0.0254	0.04226
	$\omega'B$	0.5610916	1.574738	2.62
	$\omega''B$	0.0028	0.00784	0.0131

Table 4. Frequencies (ω') and damping coefficients (ω'') of a hollow tubular bone for free vibration (first type)

		First mode	Second mode	Third mode
I	$\omega'A$	50.386734	101.08209	151.71168
	$\omega''A$	3.1082438	6.2355259	9.35075
	$\omega'B$	192.7115	386.60338	550.24372
	$\omega''B$	0.963575	1.9330169	2.7012186
II	$\omega'A$	50.260213	100.96655	151.6715
	$\omega''A$	3.10	6.228035	9.3563
	$\omega'B$	192.2276	386.13895	380.09
	$\omega''B$	0.961138	1.9306948	2.9
III	$\omega'A$	49.27242	100.53	148.82517
	$\omega''A$	3.039596	6.201616	9.2168
	$\omega'B$	188.44966	384.48936	569.2038
	$\omega''B$	0.94225	1.9225	2.84662

Table 5. Frequencies (ω') and damping coefficients (ω'') of a hollow tubular bone for clamped vibration (first type)

		First mode	Second mode	Third mode
I	$\omega'A$	50.660848	101.2482	151.80472
	$\omega''A$	3.1252501	6.2459663	9.3647809
	$\omega'B$	193.75989	387.23868	580.59954
	$\omega''B$	0.97452	1.95412	2.98425
II	$\omega'A$	50.721181	101.25183	151.82487
	$\omega''A$	3.128972	6.246189	9.366021
	$\omega'B$	193.99064	387.25258	580.67646
	$\omega''B$	0.9699532	1.9362629	2.9.33823
III	$\omega'A$	49.76445	99.35687	149.006
	$\omega''A$	3.07	0.1292856	9.192
	$\omega'B$	190.33148	380.0005	569.89786
	$\omega''B$	0.9516574	1.9	2.8495

Table 6. Frequencies (ω') and damping coefficients (ω'') of a hollow tubular bone specimen for free vibration (second type)

		First mode	Second mode	Third mode
I	$\omega'A$	27.221698	54.176553	81.190571
	$\omega''A$	1.6793122	3.3421327	5.0092633
	$\omega'B$	104.11434	207.20622	310.52532
	$\omega''B$	0.5205716	1.036031	1.5526266
II	$\omega'A$	27.155456	54.14332	81.168394
	$\omega''A$	1.6752107	3.3400827	5.00725
	$\omega'B$	103.86005	207.07912	310.4405
	$\omega''B$	0.5193002	1.0353956	1.5522025
III	$\omega'A$	1.0880899	26.560608	52.071941
	$\omega''A$	0.0671238	1.6385133	3.274
	$\omega'B$	4.1615604	101.58497	202.98147
	$\omega''B$	0.0208078	0.5079248	1.0149077

8. DISCUSSION

The predicted values of the vibration parameters for a given specimen of the vertebral body for the first three modes of vibrations (free as well as clamped) corresponding to the two different types of vibration are presented in Tables 1–3. It may be observed from these tables that the values of the parameters are distinctly different for the two different viscoelastic models studied in the paper with an aim to take into account the material damping behaviour (ascertained through experimental observations) of osseous tissues. It may be pointed out that an accurate choice of the mechanical model for an osseous medium largely depends on the correlation of experimentally measured values of the frequency and the damping coefficient with those predicted on the basis of theoretical analysis through the consideration of different analytical models for a varied range of model parameters. An investigation of the vibration characteristics of the tissues of the vertebral body may be carried out by conducting vibration experiments on a given specimen. The different sets of values presented in Tables 1–6, further indicate that material inhomogeneity of the bone specimen possesses a strong potential to considerably influence the various characteristics of vibration. This is particularly true for the case when $\alpha = 0.394$ where the elastic coefficients of osseous tissues vary nonlinearly with apparent density. It may be possible to ascertain the exact law of bone inhomogeneity for a given specimen by carrying out experimental investigations with an *a priori* knowledge of the value of the parameter α.

The values of the frequency as well as the damping coefficient computed for the two different types of vibration of a specimen of bone having hollow circularly cylindrical geometry are shown in Tables 4–6. For the first type of vibration, both free and clamped vibrations have been studied, while for the second type only the case of free vibration has been examined. As in the case of a vibrating specimen of a vertebral body, here too both the material inhomogeneity and the material damping are found to have a dominating influence over the frequency spectrum. The results presented are believed to be quite useful for the purpose of correlation with the corresponding values measured from the corresponding vibration experiments carried out with transverse sections of long bones of the skeletal system.

9. CONCLUDING REMARKS

Theoretical modelling efforts on human vertebral body response are limited by the difficulty in obtaining appropriate human test material for experimental research. Again, the simulation and prediction by analytical models of the vertebral body response to external loading behaviour requires a knowledge of the behaviour and viscoelastic mechanical properties of individual intervertebral joints. Kaleps *et al.* [31] determined the viscoelastic mechanical properties by analysis of compressive creep response data of spinal segments by Kelvin-solid models. These data are useful to explore some informations about the vertebral body mechanics. Patwardhan *et al.* [32] proposed a static simulation model, in which the equivalent motion of each intervertebral joint was also considered. Also, Quandieu and Pellieux [33] investigated the respective damping properties of the annulus fibrosis and nucleus pulposis of the intervertebral disc during propagation of vibration waves through the osteoligamento-muscular axis of the spine. Although the objects of these investigations are somewhat different, it is easy to see that there exists a good correlation between the computed values presented in this paper and experimental measurements reported by the aforementioned authors.

Acknowledgement—The authors are very grateful to the referees for their appreciation of the work and for their constructive suggestions for the improvement of the original manuscript.

REFERENCES

1. D. V. Davies and F. Davies, *Gray's Anatomy*, 33rd edn. Longmans, London (1962).
2. P. Quandieu and L. Pellieux, Study *in situ* and *in vivo* of the acceleration of lumbar vertebrae of a primate exposed to vibration in the Z-axis. *J. Biomech.* **15**, 985–1006 (1982).
3. F. G. Evans, Bones and bones. *J. biomech. Engng, Trans. Am. Soc. mech. Engrs* **104**, 1–5 (1982).

4. A. H. Soni, J. A. Sullivan Jr, A. G. Patwardhan, M. R. Gudavalli and J. U. Chitwood, Kinetic analysis and simulation of vertebral motion under static load—Part I: Kinematic analysis. *J. biomech. Engng, Trans. Am. Soc. mech. Engrs* **104**, 105–111 (1982).
5. M. Reuber, A. Schultz, F. Denis and D. Spencer, Bulging of lumbar intervertebral disks. *J. biomech. Engng, Trans. Am. Soc. mech. Engrs* **104**, 187–192 (1982).
6. H. Lin, Y. Liu and K. Adams, Mechanical response of the lumbar intervertebral joint under physiological (complex) loading. *J. Bone Jt Surg.* **60A**, 41–55 (1978).
7. H. Lin, Y. Liu, G. Ray and P. Nikravesh, Systems identification for material properties of the intervertebral joint. *J. Biomech.* **11**, 1–14 (1978).
8. A. Nachemson, A. Schultz and M. Berkson, Mechanical properties of human lumbar spine motion segments, Part III: Influences of age, sex, disc level and degeneration. *Spine* **4**, 1–18 (1979).
9. K. Broberg and H. Von Essen, Modelling of intervertebral discs. *Spine* **5**, 155–167 (1980).
10. D. Hickey and D. Hukins, Relation between the structure of the annulus fibrosis and the function and failure of the intervertebral disc. *Spine* **5**, 106–116 (1980).
11. J. H. McElhaney, Dynamic response of bone and muscle tissue. *J. appl. Physiol.* **20**, 1231–1236 (1966).
12. R. S. Lakes, J. L. Katz and S. S. Sternstein, Viscoelastic properties of wet cortical bone—I. Torsional and biaxial studies. *J. Biomech.* **12**, 657–678 (1979).
13. T. Gottesman and Z. Hashin, Analysis of viscoelastic behaviour of bones on the basis of microstructure. *J. Biomech.* **13**, 89–96 (1980).
14. R. Sanjibee, A viscoelastic model for the mechanical properties of biological materials. *J. Biomech.* **15**, 107–109 (1982).
15. R. Sanjibee, N. Somanethan and D. Ramaswamy, A. viscoelastic model for collagen fibres. *J. Biomech.* **15**, 181–183 (1982).
16. R. L. Spilker, Mechanical behaviour of a simple model of an intervertebral disc under complex loading. *J. Biomech.* **13**, 895–901 (1980).
17. R. L. Spilker, D. M. Daugirda and A. B. Schultz, Mechanical response of a simple finite element model of the intervertebral disc under complex loading. *J. Biomech.* **17**, 103–112 (1984).
18. M. L. Burns and I. Kaleps, Analysis of load-deflection behaviour of intervertebral discs under axial compression using exact parametric solutions of Kelvin-solid models. *J. Biomech.* **13**, 959–964 (1980).
19. M. L. Burns, I. Kaleps and L. E. Kazarian, Analysis of compression creep behaviour of vertebral unit subjected to a uniform axial loading using exact parametric solution equations of Kelvin-solid models. *J. Biomech.* **17**, 113–130 (1984).
20. S. B. Lang, Ultrasound method for measuring elastic coefficients of bone and results on fresh and dried bovine bones. *IEEE Trans. biomed. Engng.* **BME17**, 101–105 (1970).
21. H. S. Yoon and J. L. Katz, Ultrasonic wave propagation in human cortical bones—I. Theoretical consideration for hexagonal symmetry. *J. Biomech.* **9**, 401–412 (1978).
22. S. F. Lipson and J. L. Katz, The relationship between elastic properties and microstructure of bovine cortical bone. *J. Biomech.* **17**, 231–240 (1984).
23. J. L. Nowinski, A simulation for the stress-field in a tubular bone due to a forced-in-shaft of a prosthesis. *Int. J. mech. Sci.* **16**, 285–288 (1974).
24. M. R. Patel, The deformation and fracture of rigid cellar plastics under multiaxial stress. Ph.D. Thesis. University of California, Berkeley (1969).
25. D. R. Carter, W. C. Hayes and D. J. Schruman, Fatigue life of compact bone—Part II: Effects of microstructure and density. *J. Biomech.* **9**, 211–218 (1976).
26. J. L. Stone, G. S. Beaupre and W. C. Hayes, Bone compressive strength. *J. Biomech.* **15**, 743–752 (1982).
27. J. C. Misra and S. Chakravarty, A free-vibration analysis for the human cranial system. *J. Biomech.* **15**, 635–645 (1982).
28. J. C. Misra and S. C. Samanta, Propagation of torsional waves in tubular bones. *J. math. Analysis Applic.* **96**, 313–329 (1983).
29. J. C. Misra and S. C. Samanta, Wave propagation in tubular bones. *Int. J. Solids Struct.* **20**, 55–62 (1984).
30. J. C. Misra and S. C. Samanta, Effect of material damping on bone remodelling. *J. Biomech.* **20**, 241–251 (1987).
31. I. Kaleps, L. E. Kazarian and M. L. Burns, Analysis of compressive creep behaviour of the vertebral unit subjected to a uniform axial loading using exact parametric solution equations of Kelvin-solid models—Part II. Rhesus monkey intervertebral joints. *J. Biomech.* **17**, 131–136 (1984).
32. A. G. Patwardhan, A. H. Soni, J. A. Sullivan Jr, M. R. Gudavalli and V. Srinivasan, Kinematic analysis and simulation of the vertebral motion under static load—Part II: Simulation study. *J. biomech. Engng, Trans. Am. Soc. mech. Engrs* **104**, 112–118 (1982).
33. P. Quandieu and L. Pellieux, Effects of the ablation of the nucleus pulposus on the vibrational behaviour of the lumbosacral hinge. *J. Biomech.* **16**, 777–784 (1983).